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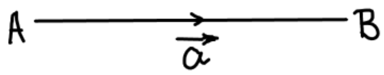
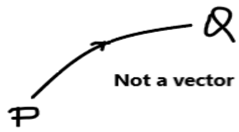
Topic for—Paper MTMA CC-1 (Vector Analysis)

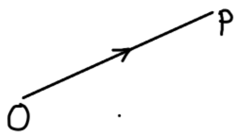
Vector Analysis: Application to Geometry and Mechanics

Vector: A directed line segment is a vector quantity. If the vector $\overrightarrow{AB}$ is denoted by $\vec{a}$, then its magnitude (i.e. length) is denoted by $|\overrightarrow{AB}| = |\vec{a}|$.





Position Vector of a point: $\overrightarrow{OP}$ is the position vector of the point P with respect to the point O. Similarly, $\overrightarrow{PO}$ is the position vector of the point O with respect to the point P.



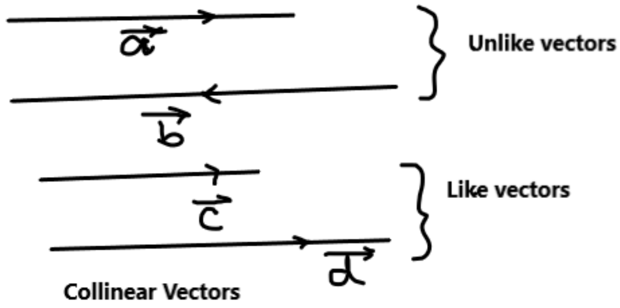
Note: $\overrightarrow{AB} =$ Position vector of the point **B** – Position vector of the point **A.**

Free vector: It can be shifted to any position in space keeping its length and direction unchanged.

Localised vector: It has a fixed line of support. It cannot be shifted to any position. Force vector is a localised vector as it depends upon position and direction.

Unit vector: vector having magnitude(length) a unit. Unit vector of $\vec{a}$ is denoted by $\hat{a}$ and is defined by $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$. Thus, $\vec{a} = |\vec{a}|\hat{a}$

Collinear vectors: Vectors parallel to a fixed line (not necessarily lying on the same line). In this case, vectors having the same direction are known as **like vectors** and having the opposite direction with respect to each other are termed to be **unlike vectors**.



Theorem: If the two non-zero vectors are collinear, then one can be expressed as a scalar multiple of the other. i.e. if $\vec{a}$ and $\vec{b}$ are collinear, then

$$\vec{\boldsymbol{a}} = \boldsymbol{k}\vec{\boldsymbol{b}},\ \ k = \text{some scalar.}$$

Coplanar vectors: Vectors parallel to a fixed plane.

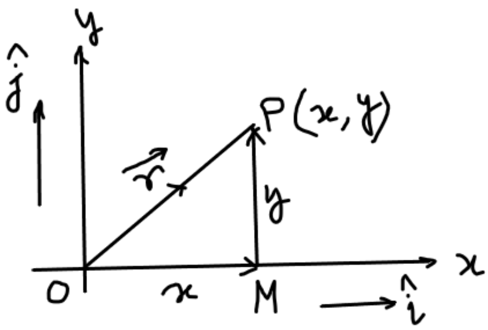
Note: Collinear vectors are always coplanar but the converse is not true.

Resolution of a vector: Let us take unit vector along x-axis as $\hat{\imath}$ and along y-axis as $\hat{\jmath}$. If a point P on a plane has coordinates (x, y), then abscissa of P is OM = x and ordinate of P is MP = y. Then $\overrightarrow{OM} = x\hat{\imath}$ and $\overrightarrow{MP} = y\hat{\jmath}$ [as $|\overrightarrow{OM}| = x,\ |\overrightarrow{MP}| = y$].

Let $\overrightarrow{OP} = \vec{r}$. From the law of vector addition, $\overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MP}$

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} \quad \text{and} \quad |\vec{r}| = \sqrt{x^2 + y^2}$$

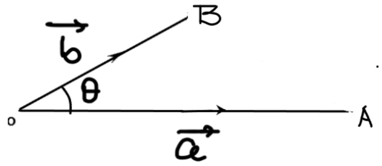
The unit vector along $\vec{r}$ is $\hat{r} = \left(\frac{x}{r}\right)\hat{\imath} + \left(\frac{y}{r}\right)\hat{\jmath}$



In three dimensions, if a point P has coordinates (x, y, z) and $\overrightarrow{OP} = \vec{r}$, then $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ where $\hat{\imath},\ \hat{\jmath},\ \hat{k}$ are the unit vectors along x-axis, y-axis and z-axis respectively and

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2}.$$

Scalar Product or dot Productor: $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$ where θ $(0 \le \theta \le \pi)$ is the angle between the direction of $\vec{a}$ & $\vec{b}$.



Properties: i) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

ii) $\vec{a} \cdot \vec{a} = |\vec{a}|^2 = \vec{a}^2$

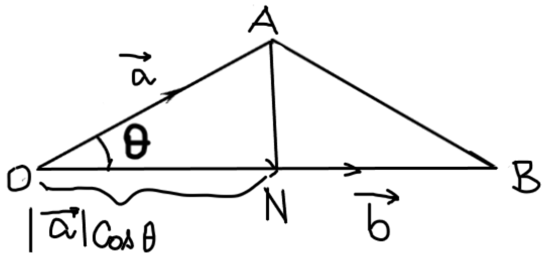
iii) If $\vec{a}$ & $\vec{b}$ are mutually perpendicular, then $\vec{a} \cdot \vec{b} = 0$

iv) If $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ & $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$, then

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Component of a vector $\vec{a}$ along a vector $\vec{b}$ (Projection of $\vec{a}$ along $\vec{b}$) is $\overrightarrow{ON}$ which is

$$\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}\right)\hat{b} \quad or \quad \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2}\right)\vec{b}$$



Component of a vector $\vec{a}$ perpendicular to a vector $\vec{b}$ is $\overrightarrow{NA}$ (= $\overrightarrow{NO} + \overrightarrow{OA}$)

$$\vec{a} - \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}\right)\hat{b} \quad or \quad \vec{a} - \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2}\right)\vec{b}$$

Vector Product or cross Product: $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\ \hat{n}$ where θ $(0 \le \theta \le \pi)$ is the angle between the direction of $\vec{a}$ & $\vec{b}$ and $\hat{n}$ is the unit vector perpendicular to the plane of $\vec{a}$ & $\vec{b}$ and in the direction of translation of a right-handed screw due to the rotation from $\vec{a}$ to $\vec{b}$.

Properties: i) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

ii) If two vectors $\vec{a}$ and $\vec{b}$ are collinear, then $\vec{a} \times \vec{b} = 0$

iii) If $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ & $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$, then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

iv) Unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is $\pm \left(\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}\right)$

Scalar Triple Product: $[\vec{\boldsymbol{a}}\ \vec{\boldsymbol{b}}\ \vec{\boldsymbol{c}}] = \vec{\boldsymbol{a}} \cdot (\vec{b} \times \vec{c})$. It is a scalar quantity.

Properties:

i) Numerical value of $\vec{\boldsymbol{a}} \cdot (\vec{b} \times \vec{c})$ represents the **volume of a parallelopiped** having $\vec{a}, \vec{b}, \vec{c}$ as concurrent edges. Its sign will be positive or negative according as the vectors $\vec{a}, \vec{b}, \vec{c}$ form a right-handed system or a left-handed system.

ii) Three non-zero, non-collinear vectors $\vec{a},\ \vec{b},\ \vec{c}$ are coplanar if and only if $[\vec{\boldsymbol{a}}\ \vec{\boldsymbol{b}}\ \vec{\boldsymbol{c}}] = \vec{\boldsymbol{0}}$

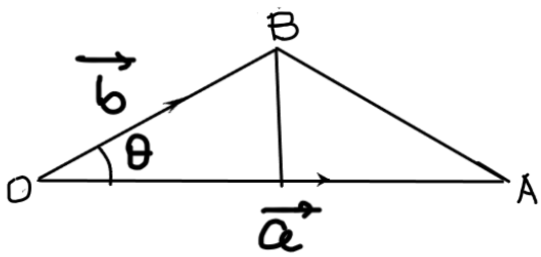
iii) If $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ & $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ & $\vec{c} = c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{k}$ then

$$[\vec{\boldsymbol{a}}\ \vec{\boldsymbol{b}}\ \vec{\boldsymbol{c}}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Application to Geometry:

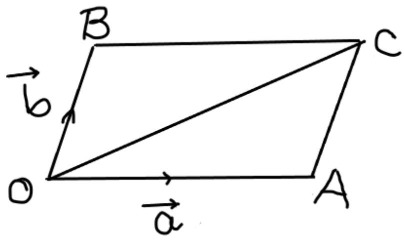
1. **Area of a triangle:** Area of a triangle OAB, where $\overrightarrow{OA} = \vec{a},\ \overrightarrow{OB} = \vec{b}$ is $\frac{1}{2}|\vec{a} \times \vec{b}|$

Vector area of triangle OAB is $\frac{1}{2}(\vec{a} \times \vec{b})$.

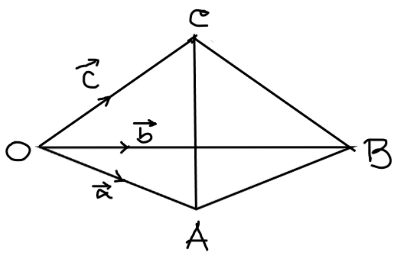


2. **Area of a parallelogram:** Area of a parallelogram OACB, where $\overrightarrow{OA} = \vec{a},\ \overrightarrow{OB} = \vec{b}$ is $|\vec{a} \times \vec{b}|$

Vector area of parallelogram OACB is $(\vec{a} \times \vec{b})$.

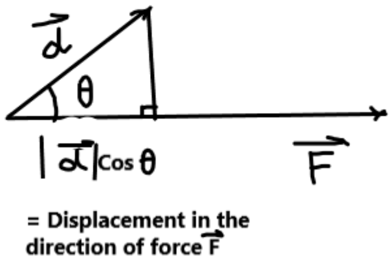


3. **Volume of a tetrahedron:** Volume of a tetrahedron OACB having triangle OAB as its base and C as fourth vertex where $\overrightarrow{OA} = \vec{a},\ \overrightarrow{OB} = \vec{b},\ \overrightarrow{OC} = \vec{c}$ is $\frac{\mathbf{1}}{\mathbf{6}}\ [\vec{\boldsymbol{a}}\ \vec{\boldsymbol{b}}\ \vec{\boldsymbol{c}}]$



Application to Mechanics:

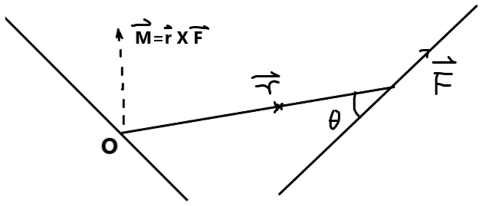
1. Work done by a Force:



$\vec{\boldsymbol{F}}$ = given force
$\vec{\boldsymbol{d}}$ = displacement
Then, work done **W** by the force is given by $\boldsymbol{W} = \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{d}}$
(i.e. $W = |\vec{\boldsymbol{F}}||\vec{\boldsymbol{d}}| \cos\theta$)

Note: Work done will be **Maximum** if $\vec{F}$ and $\vec{d}$ are in the same direction and work done will be **Minimum** if $\vec{F}$ and $\vec{d}$ are in the opposite directions. If the force $\vec{F}$ and the displacement $\vec{d}$ are mutually perpendicular, then **work done will be zero**.

2. Torque or Moment of a force about a point:



$\vec{\boldsymbol{F}}$ = given force (a localised vector)

O = given point about which moment of $\vec{\boldsymbol{F}}$ is to be found

$\vec{\boldsymbol{r}}$ =position vector of any point on the line of action of $\vec{\boldsymbol{F}}$

Then, Moment of $\vec{\boldsymbol{F}}$ about the point O is $\overrightarrow{\boldsymbol{M}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}}$

Examples:

1. Find the torque about the point (1, 1, 1) of a force of magnitude 15 units acting at a point (2, -2, 2) in the direction of the vector $\hat{\imath} - 2\hat{\jmath} + 2\hat{k}$.

Solution: Let $\vec{F}$ be the given force. Then $|\vec{F}|$= 15 units.

Unit vector in the direction of the vector $\hat{\imath} - 2\hat{\jmath} + 2\hat{k}$ is $\frac{\hat{\imath}-2\hat{\jmath}+2\hat{k}}{\sqrt{1+4+4}} = \frac{1}{3}\left(\hat{\imath} - 2\hat{\jmath} + 2\hat{k}\right)$.

$$\therefore \quad \vec{F} = 15\ \frac{1}{3}\left(\hat{\imath} - 2\hat{\jmath} + 2\hat{k}\right) = 5(\hat{\imath} - 2\hat{\jmath} + 2\hat{k})$$

P(2, -2, 2) is a point on the line of action of the force $\vec{F}$

Then, Position vector of P relative to the given point A(1, 1, 1) is

$$\vec{r} = \overrightarrow{AP} = (2,-2,2) - (1,1,1) = (1,-3,1) = \hat{\imath} - 3\hat{\jmath} + \hat{k}$$

The Torque about the point A of the force $\vec{F}$ is

$$\overrightarrow{\boldsymbol{M}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}} = \mathbf{5}\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -3 & 1 \\ 1 & -2 & 2 \end{vmatrix} = -\mathbf{20}\hat{\imath} - 5\hat{\jmath} + 5\hat{k}$$

Magnitude of the torque $|\overrightarrow{\boldsymbol{M}}| = \sqrt{400 + 25 + 25} = 15\sqrt{2}$ units.

2. A particle being acted on by constant forces $4\hat{\imath}+\hat{\jmath}-3\hat{k}$ and $\widehat{3\imath}+\hat{\jmath}-\hat{k}$ is displaced from the point (1, 2, 3) to (5, 4, -1). Find the total work done.

Solution: Let $\overrightarrow{F_1}=4\hat{\imath}+\hat{\jmath}-3\hat{k}$ and $\overrightarrow{F_2}=\widehat{3\imath}+\hat{\jmath}-\hat{k}$

Displacement of the particle from A(1, 2, 3) to B(5, 4, -1) is

$$\vec{d}=\overrightarrow{AB}=(5,4,-1)-(1,2,3)=(4,2,-4)=4\hat{\imath}+\widehat{2\jmath}-4\hat{k}$$

Then, total work done by the forces $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$ is

$W=\overrightarrow{F_1}\cdot\vec{d}+\overrightarrow{F_2}\cdot\vec{d}=(16+2+12)+(12+2+4)=48\,units.$

3. Find the volume of the tetrahedron with vertices P (−1,2,0), Q (2,1, −3), R 1,0,1) and S (3, −2,3).

Solution:

Given vertices of the tetrahedron are P(−1, 2, 0), Q(2, 1, −3), R(1, 0, 1) and S(3, −2, 3)

The volume of a tetrahedron is equal to $\frac{1}{6}$ of the absolute value of the triple product.

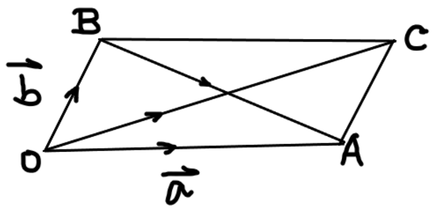
$$V=\frac{1}{6}[\vec{PQ}\quad \vec{PR}\quad \vec{PS}]$$

$$\rightarrow \vec{PQ}=3i-j-3k;\ \vec{PR}=2i-2j+k \quad \text{and} \quad \vec{PS}=4i-4j+3k$$

$$\Rightarrow V=\frac{1}{6}\begin{vmatrix}3 & -1 & -3\\ 2 & -2 & 1\\ 4 & -4 & 3\end{vmatrix}$$

$$\therefore V=\frac{2}{3}$$

4. If the diagonals of a parallelogram are equal then show by vector method that it is a rectangle.
Solution:



Let OACB be a parallelogram where $\overrightarrow{OA} = \vec{a} \;\; and \;\; \overrightarrow{OB} = \vec{b}$. Therefore, the diagonals are represented by the vectors $\overrightarrow{OC} = \vec{a} + \vec{b} \;\; and \;\; \overrightarrow{BA} = \vec{a} - \vec{b}$.
According to the problem, the diagonals are equal. i.e., OC = BA.

$\therefore \;\; |\overrightarrow{OC}| = |\overrightarrow{BA}|$ or $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$

or, $|\vec{a} + \vec{b}|^2 = |\vec{a} - \vec{b}|^2$

or, $(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b})$

or, $|\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a} \cdot \vec{b} = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a} \cdot \vec{b}$

or, $\overrightarrow{4a} \cdot \vec{b} = 0$

or, $\vec{a} \cdot \vec{b} = 0$

Therefore, $\vec{a}$ and $\vec{b}$ are perpendicular.
i.e., angle between vector $\overrightarrow{OA}$ and vector $\overrightarrow{OB}$ is a right angle.
i.e.,

$$\angle BOA = 90^{0}$$

Hence, OABC is a rectangle.